Robust probability updating

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UvA

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Monty Hall’s game show

Initial probability:

\begin{align*}
\text{Door 1}: & \quad \frac{1}{3} \\
\text{Door 2}: & \quad \frac{1}{3} \\
\text{Door 3}: & \quad \frac{1}{3}
\end{align*}
Monty Hall’s game show

Initial probability:

1/3  1/3  1/3
Monty Hall’s game show

Initial probability:

1/3 1/3 1/3

New probability:

? ? 0
Step 1 **Outcome** $X$ is randomly drawn from $\mathcal{X} = \{x_1, x_2, x_3\}$ (the three doors) according to the uniform distribution $p$
Formalizing the problem

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Step 2 The quizmaster, knowing $X$, chooses a set $Y \in \mathcal{Y} = \{\{x_1, x_2\}, \{x_2, x_3\}\}$ such that $Y \ni X$

- The structure of $\mathcal{Y}$ reflects that the quizmaster will always open one door, but never the door the contestant picked
- The chosen set $Y$ is called the **message**
Formalizing the problem

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**Step 3** The contestant sees $Y$ but not $X$, and responds with a prediction $Q$: his estimate of the correct probability distribution over $\mathcal{X}$
Generalizing the problem

We also want to know what probabilities to assign to the outcomes in a more general situation:

- For arbitrary (but finite) outcome spaces $\mathcal{X}$;
- For arbitrary marginal distribution $p$;
- For arbitrary families of allowed messages $\mathcal{Y}$. 
The quizmaster’s freedom of choice

- The quizmaster may use randomness when deciding which message $Y$ to give us
- However, we don’t know what distribution $P(Y \mid X)$ he uses
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- However, we don’t know what distribution $P(Y \mid X)$ he uses.
- The conditional distribution $P$ together with the marginal distribution $p$ on $\mathcal{X}$ gives a joint distribution:

<table>
<thead>
<tr>
<th>Quizmaster uses fair coin:</th>
<th>Quizmaster always opens $x_3$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>${x_1, x_2}$</td>
<td>$1/3$</td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>$-$</td>
</tr>
<tr>
<td>$p_x$</td>
<td>$1/3$</td>
</tr>
</tbody>
</table>
Worst-case approach

- We don’t want to assume anything about how the quizmaster decides what message to give us
  - In the case of the original Monty Hall game, the fair-coin assumption can be defended based on the symmetry of the problem, but there may not be any symmetry in the general case
- Worst-case approach: we want to give guarantees on our predictions that hold no matter what mechanism is used to choose the message
- Corresponds to a two-player zero-sum game between the contestant and the quizmaster, where the quizmaster tries to make the contestant’s prediction task as hard as possible
Loss functions

To do this, we need some way of quantifying how good a prediction is.

We use a **loss function** that maps actual outcome $x$ and contestant’s prediction $Q$ to some loss value for the contestant.

Examples:

- **Logarithmic loss:**
  \[ L(x, Q) = -\log Q(x) \]

- **Brier loss:**
  \[ L(x, Q) = \sum_{x' \in X} (Q(x') - 1_{x' = x}) \]

- **Randomized 0–1 loss:**
  \[ L(x, Q) = 1 - Q(x) \]
The value of the game is

$$\sum_{x \in X} \sum_{y \in Y} P(x, y) L(x, Q_y)$$

The contestant chooses \((Q_y)_{y \in Y}\) to minimize this; the quizmaster chooses \(P\) to maximize this.

For many loss functions, this game has a Nash equilibrium: there exists a pair of strategies for both players such that neither player would improve their situation by changing their strategy.
We proved optimality conditions for a very general class of loss functions.

For the special case that $L$ is logarithmic loss, the characterization of optimality takes a very nice form:

**Theorem**

*For logarithmic loss, a joint distribution $P^*$ is optimal for the quizmaster if and only if there exists a vector $q \in [0, 1]^X$ such that*

$$q_x = P^*(x \mid y) \text{ for all } x \in y \in \mathcal{Y} \text{ with } P^*(y) > 0, \text{ and}$$

$$\sum_{x \in y} q_x \leq 1 \text{ for all } y \in \mathcal{Y}$$

We call this condition on $P^*$ the **RCAR condition**.
Optimal strategy may depend on the loss function

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1, x_2}$</td>
<td>$1/3$</td>
<td>$1/6$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>$-$</td>
<td>$1/6$</td>
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</table>

This strategy $P$ is optimal for logarithmic loss (it satisfies the RCAR condition), but not for Brier loss.
If the set of available messages $\mathcal{Y}$ forms a graph (meaning that each message contains exactly two outcomes), then the RCAR condition characterizes optimality regardless of the loss function;

If $\mathcal{Y}$ forms a matroid (satisfies the matroid basis exchange property), then the same is true;

For any other $\mathcal{Y}$, this is not the case: there exists some marginal $p$ such that the optimal strategies for log loss and Brier loss are different.
Worst-case optimal probability updating gives us a general framework that we can use to update our beliefs about an outcome in situations where the joint distribution of outcome and evidence is unknown.

In general, the answer depends on the loss function ("subjective"), but in certain situations they don’t ("objective").